$$\nabla^2 \phi - \frac{1}{L^2}\phi + \frac{Q}{D}\delta(x)\delta(y)\delta(z) = 0 \qquad \lim_{x,y,z\to\pm\infty} \phi = 0$$

Trasformiamo secondo Fourier 3 volte (x, y e z)

$$-(\omega_x^2 + \omega_y^2 + \omega_z^2)\,\psi - \frac{1}{L^2}\psi(\omega_x, \omega_y, \omega_z) + Q_0 = 0$$

$$\Downarrow$$

$$\psi = \frac{Q_0}{\omega_x^2 + \omega_y^2 + \omega_z^2 + \frac{1}{L^2}} \qquad \left[Q_0 = \frac{Q}{D} \right]$$

ovvero, detto $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$:

$$\psi = \frac{Q_0}{\omega^2 + \frac{1}{L^2}}$$

Dobbiamo ora antitrasformare:

$$\phi = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\omega_x x} \left\{ \int_{-\infty}^{\infty} e^{i\omega_y y} \left[\int_{-\infty}^{\infty} e^{i\omega_z z} \psi \, d\omega_z \right] d\omega_y \right\} d\omega_x$$

$$= \frac{1}{(2\pi)^3} \iiint e^{i[\omega_x x + \omega_y y + \omega_z z]} \psi \, d\omega_x \, d\omega_y \, d\omega_z$$

Poniamo $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\phi = \frac{1}{(2\pi)^3} \iiint e^{i\vec{\omega}\cdot\vec{r}} \psi \, d\omega_x \, d\omega_y \, d\omega_z$$

Cambiamo sistema di coordinate e istituiamo coord. sferiche (per $\vec{\omega}$) aventi come asse polare $\vec{r}$

$$\phi = \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{i\omega r\cos\theta} \frac{Q_0}{\omega^2 + \frac{1}{L^2}} \omega^2 \operatorname{sen}\theta \, d\omega \, d\theta \, d\varphi$$

$$= \frac{Q_0}{(2\pi)^2} \int_0^\infty \int_{-1}^1 \frac{e^{i\omega r\mu}}{\omega^2 + \frac{1}{L^2}} \omega^2 \, d\omega \, d\mu =$$

$$= \frac{Q_0}{(2\pi)^2} \int_0^\infty \frac{\omega^2}{\omega^2 + \frac{1}{L^2}} \left\{ \int_{-1}^1 e^{i\omega r\mu} d\mu \right\} d\omega$$

$$= \frac{Q_0}{(2\pi)^2} \int_0^\infty \frac{\omega^2}{\omega^2 + \frac{1}{L^2}} \left\{ \frac{e^{i\omega r} - e^{-i\omega r}}{i\omega r} \right\} d\omega$$

$$= \frac{Q_0}{(2\pi)^2} \frac{1}{ir} \int_0^\infty \frac{\omega \, d\omega \, (e^{i\omega r} - e^{-i\omega r})}{\omega^2 + \frac{1}{L^2}} d\omega$$

$$= \frac{Q_0}{(2\pi)^2 i r} \left\{ \int_0^\infty \frac{\omega e^{i\omega r}}{\omega^2 + \frac{1}{L^2}} d\omega - \int_0^\infty \frac{\omega e^{-i\omega r}}{\omega^2 + \frac{1}{L^2}} d\omega \right\} =$$

$$= \frac{Q_0}{(2\pi)^2 i r} \left\{ \int_0^\infty \frac{\omega e^{i\omega r}}{\omega^2 + \frac{1}{L^2}} d\omega + \int_{-\infty}^0 \frac{\omega e^{i\omega r}}{\omega^2 + \frac{1}{L^2}} d\omega \right\} =$$

$$= \frac{Q_0}{(2\pi)^2 i r} \int_{-\infty}^\infty \frac{\omega e^{i\omega r}}{\omega^2 + \frac{1}{L^2}} d\omega$$

la funzione ha due singolarità, in $\omega = \pm i \frac{1}{L}$

Applichiamo il metodo dei residui:

$$= \frac{Q_0}{(2\pi)^{\cancel{2}} \cancel{i} r} \cdot \cancel{2\pi i} \, \mathrm{Res}_{\omega = \frac{i}{L}} \left[\frac{\omega e^{i\omega r}}{\omega^2 + \frac{1}{L^2}} \right]$$

$$\mathrm{Res} = \lim_{\omega \to i\frac{1}{L}} \frac{\omega e^{i\omega r}}{\omega + i\frac{1}{L}} = \frac{e^{-r/L}}{2}$$

$$\phi = \frac{Q}{4\pi D} \frac{e^{-r/L}}{r}$$