POLARIZATION STATE → WAVE NATURE OF PHOTONS

FACTS

- Photons obey polarization dependent interactions.
- The models describing photon transport usually ignore polarization or treat it in incomplete way

QUESTIONS

- Are different the intensities predicted with a vector model and with a scalar model?
- Which is the extent of the difference?
- How it depends on the polarization of the source?
- How it depends on the type and the number of collisions involved?

Polarization

Some definitions :

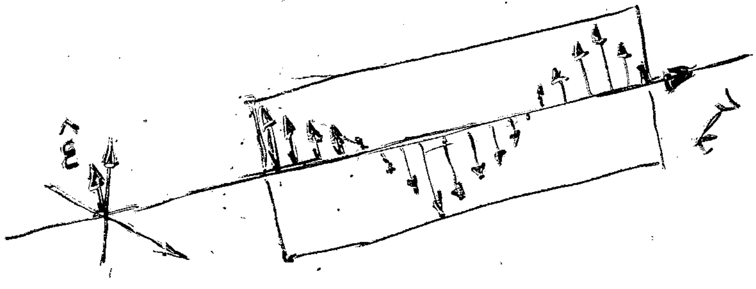
- Plane waves have the electric field $\vec{E}$ lying in a perpendicular plane to the direction of propagation $\vec{k}$.

$$\Rightarrow \quad \vec{E} \cdot \vec{k} = 0$$

- A plane wave having the electric field $\vec{E}$ always in the same direction $\hat{\varepsilon}$ is said to be linearly polarized.

$$\Rightarrow \quad \vec{E} = \hat{\varepsilon} \, E_0 \, e^{i\vec{k}\cdot\vec{x} - i\omega t}$$ (plane wave with linear polarization)

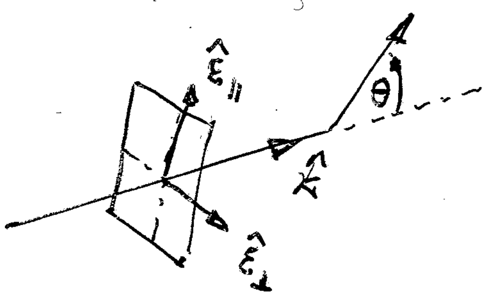
$\hat{\varepsilon}$ ⇒ fix !!



- A general state of polarization can be described with two (independent) waves linearly polarized

$$\Longrightarrow \quad \vec{E} = (\hat{\varepsilon}_1 E_1 + \hat{\varepsilon}_2 E_2)\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

We choose the directions $\hat{\varepsilon}_{\parallel}$ and $\hat{\varepsilon}_{\perp}$ (perpendicular) to decompose the electric field vector



the scattering defines a plane and $\hat{\varepsilon}_{\parallel}$ and $\hat{\varepsilon}_{\perp}$ are parallel and perpendicular to this plane

$$\Longrightarrow \quad \vec{E} = E_{\parallel}\, \hat{\varepsilon}_{\parallel} + E_{\perp}\, \hat{\varepsilon}_{\perp}$$

in general we can write

$$\begin{cases} E_{\parallel} = E^{0}_{\parallel} \sin(\omega t - \delta_{\parallel}) & \text{(P.1)} \\ E_{\perp} = E^{0}_{\perp} \sin(\omega t - \delta_{\perp}) & \text{(P.2)} \end{cases}$$

Since the intensity is proportional to the square of the electric vector we define

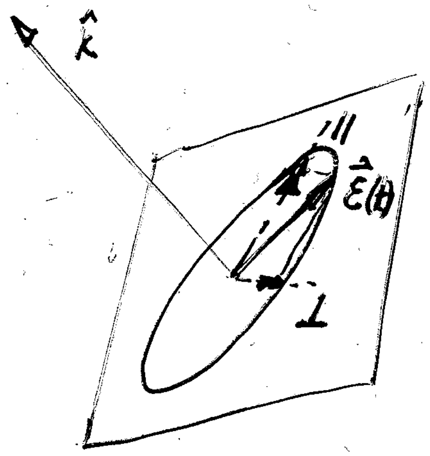
$$I = (E^0_{\parallel})^2 + (E^0_{\perp})^2 = I_{\parallel} + I_{\perp}$$

Eqns (P.1) and (P.2) can be written as a unique parametric equation if time is eliminated

$$\left[\frac{E_{\parallel}(t)}{E^0_{\parallel}}\right]^2 - 2\cos(\delta_{\parallel}-\delta_{\perp})\frac{E_{\parallel}(t)\,E_{\perp}(t)}{E^0_{\parallel}\,E^0_{\perp}} + \left[\frac{E_{\perp}(t)}{E^0_{\perp}}\right]^2 = \sin^2(\delta_{\parallel}-\delta_{\perp})$$

this term makes that the axes of this ellipse do not coincide with the $\hat{\varepsilon}_{\parallel}$ and $\hat{\varepsilon}_{\perp}$ axes

REPRESENTATION OF POLARIZED LIGHT



components of the electric vector

$$\mathcal{E}_{\parallel}(t) = \mathcal{E}^0_{\parallel} \sin(\omega t - \delta_{\parallel})$$
$$\mathcal{E}_{\perp}(t) = \mathcal{E}^0_{\perp} \sin(\omega t - \delta_{\perp})$$

ω the frequency

Since the intensity is proportional to the square of the electric field, then we can choose

$$I = (\mathcal{E}^0_{\parallel})^2 + (\mathcal{E}^0_{\perp})^2 = I_{\parallel} + I_{\perp}$$

$$I \propto \mathcal{E}^2(t)$$

By averaging in one period we get

$$\int_0^T dt\, I \propto \int_0^T dt\, [\mathcal{E}^2_{\parallel}(t) + \mathcal{E}^2_{\perp}(t)]$$

By eliminating the time variable in the above equations we obtain

$$\left(\frac{\mathcal{E}_{\parallel}(t)}{\mathcal{E}^0_{\parallel}}\right)^2 - 2\cos(\delta_{\parallel} - \delta_{\perp}) \frac{\mathcal{E}_{\parallel}(t)\,\mathcal{E}_{\perp}(t)}{\mathcal{E}^0_{\parallel}\,\mathcal{E}^0_{\perp}} + \left(\frac{\mathcal{E}_{\perp}(t)}{\mathcal{E}^0_{\perp}}\right)^2 = \sin^2(\delta_{\parallel} - \delta_{\perp})$$

(equation of an ellipse)

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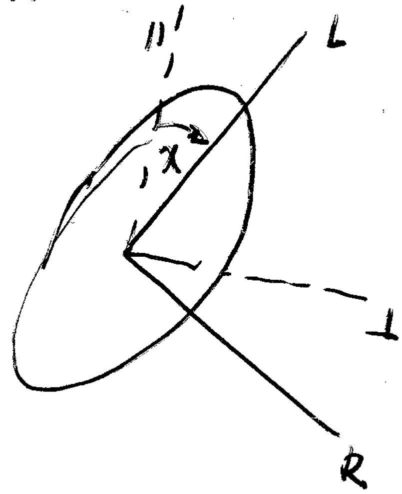
In general, the axes of this ellipse do not coincide with those $\parallel$ and $\perp$. This is expressed by the cross term, which is different from zero.

To find the axes of the ellipse we should perform a rotation χ



$$\mathcal{E}_{\parallel} = \mathcal{E}_L \cos\chi - \mathcal{E}_R \sin\chi$$
$$\mathcal{E}_{\perp} = \mathcal{E}_L \sin\chi + \mathcal{E}_R \cos\chi$$

By replacing into the cross product term we get

$$\left[\frac{1}{\mathcal{E}_{\perp}^{o\,2}} - \frac{1}{\mathcal{E}_{\parallel}^{o\,2}}\right] \sin 2\chi - 2\,\frac{\cos(\delta_{\parallel}-\delta_{\perp})}{\mathcal{E}_{\parallel}^{o}\,\mathcal{E}_{\perp}^{o}} \cos 2\chi = 0$$

from which

$$\boxed{tg\, 2\chi = \frac{2\,\mathcal{E}_{\perp}^{o}\,\mathcal{E}_{\parallel}^{o} \cos(\delta_{\parallel}-\delta_{\perp})}{(\mathcal{E}_{\parallel}^{o})^2 - (\mathcal{E}_{\perp}^{o})^2}}$$

The other quantity of interest is the ellipticity, i.e. the ratio of the axes of the ellipse

From the substitution in the ellipse equation, we get

$$tg^2\rho = \frac{\dfrac{\cos^2\chi}{(\mathcal{E}_{\parallel}^{o})^2} - \dfrac{\sin 2\chi}{\mathcal{E}_{\parallel}^{o}\mathcal{E}_{\perp}^{o}} \cos(\mathcal{E}_{\parallel}-\mathcal{E}_{\perp}) + \dfrac{\sin^2\chi}{(\mathcal{E}_{\perp}^{o})^2}}{\sin^2\chi + \sin 2\chi \cos(\mathcal{E}_{\parallel}-\mathcal{E}_{\perp}) + \cos^2\chi}$$

and after some algebra

$$\sin 2\beta = \pm \frac{2\,\mathcal{E}_{\parallel}^{0}\,\mathcal{E}_{\perp}^{0}\sin(\delta_{\parallel}-\delta_{\perp})}{(\mathcal{E}_{\parallel}^{0})^2+(\mathcal{E}_{\perp}^{0})^2}$$

The Stokes parameters are defined as

$$I = (\mathcal{E}_{\parallel}^{0})^2+(\mathcal{E}_{\perp}^{0})^2 = I_{\parallel}+I_{\perp}$$

$$Q = (\mathcal{E}_{\parallel}^{0}) - (\mathcal{E}_{\perp}^{0})^2 = I_{\parallel} - I_{\perp}$$

$$U = 2\,\mathcal{E}_{\parallel}^{0}\,\mathcal{E}_{\perp}^{0}\cos(\delta_{\parallel}-\delta_{\perp})$$

$$V = 2\,\mathcal{E}_{\parallel}^{0}\,\mathcal{E}_{\perp}^{0}\sin(\delta_{\parallel}-\delta_{\perp})$$

From the above equations

$$\cos 2\beta = \pm\frac{\left\{\left[(\mathcal{E}_{\parallel}^{0})^2-(\mathcal{E}_{\perp}^{0})^2\right]^2+4\,\mathcal{E}_{\parallel}^{0\,2}\mathcal{E}_{\perp}^{0\,2}\cos^2(\delta_{\parallel}-\delta_{\perp})\right\}^{1/2}}{(\mathcal{E}_{\parallel}^{0})^2+(\mathcal{E}_{\perp}^{0})^2}$$

$$\sin 2\chi = \pm\frac{2\,\mathcal{E}_{\parallel}^{0}\,\mathcal{E}_{\perp}^{0}\cos(\delta_{\parallel}-\delta_{\perp})}{\left\{\left[(\mathcal{E}_{\parallel}^{0})^2-(\mathcal{E}_{\perp}^{0})^2\right]^2+4\,\mathcal{E}_{\parallel}^{0\,2}\mathcal{E}_{\perp}^{0\,2}\cos^2(\delta_{\parallel}-\delta_{\perp})\right\}^{1/2}}$$

$$\cos 2\chi = \pm\frac{(\mathcal{E}_{\parallel}^{0})^2-(\mathcal{E}_{\perp}^{0})^2}{\left\{\left[(\mathcal{E}_{\parallel}^{0})^2-(\mathcal{E}_{\perp}^{0})^2\right]^2+4\,\mathcal{E}_{\parallel}^{0\,2}\mathcal{E}_{\perp}^{0\,2}\cos(\delta_{\parallel}-\delta_{\perp})\right\}^{1/2}}$$

by replacing above, the Stokes parameters become

$$Q = I\cos 2\beta\cos 2\chi$$

$$U = I\cos 2\beta\sin 2\chi$$

$$V = I\sin 2\beta$$

BOLTZMANN TRANSPORT EQUATION FOR POLARIZED PHOTONS

(Chandrasekhar)

$$\vec{\omega}\cdot\nabla\vec{f}(\vec{r},\vec{\omega},\lambda) = -\mu(\vec{r},\lambda)\,\vec{f}(\vec{r},\vec{\omega},\lambda) + \int_0^\infty d\lambda' \int_{4\pi} d\vec{\omega}'\, \mathbb{H}(\vec{\omega},\lambda,\vec{\omega}';\lambda')\,\vec{f}(\vec{r},\vec{\omega}',\lambda') + \vec{S}(\vec{r},\vec{\omega},\lambda)$$

where

$$\vec{f} = \begin{pmatrix} f_I \\ f_Q \\ f_U \\ f_V \end{pmatrix} \qquad \vec{S} = \begin{pmatrix} S_I \\ S_Q \\ S_U \\ S_V \end{pmatrix}$$

$$\mathbb{H}(\vec{\omega},\lambda,\vec{\omega}';\lambda') = \mathbb{L}(\pi-\psi)\,\underbrace{\mathbb{K}(\omega,\lambda,\vec{\omega}';\lambda')}_{\text{Scattering matrix}}\,\mathbb{L}(-\psi')$$

$$\cos\psi = \frac{\eta'\sqrt{1-\eta^2} - \eta\sqrt{1-\eta'^2}\cos(\varphi-\varphi')}{\left[1-(\vec{\omega}'\cdot\vec{\omega})^2\right]^{1/2}}$$

$$\cos\psi' = \frac{\eta\sqrt{1-\eta'^2} - \eta'\sqrt{1-\eta^2}\cos(\varphi-\varphi')}{\left[1-(\vec{\omega}'\cdot\vec{\omega})^2\right]^{1/2}}$$

$$\mathbb{L}(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Mäurdy / Bubbell / Hanson / Spencer

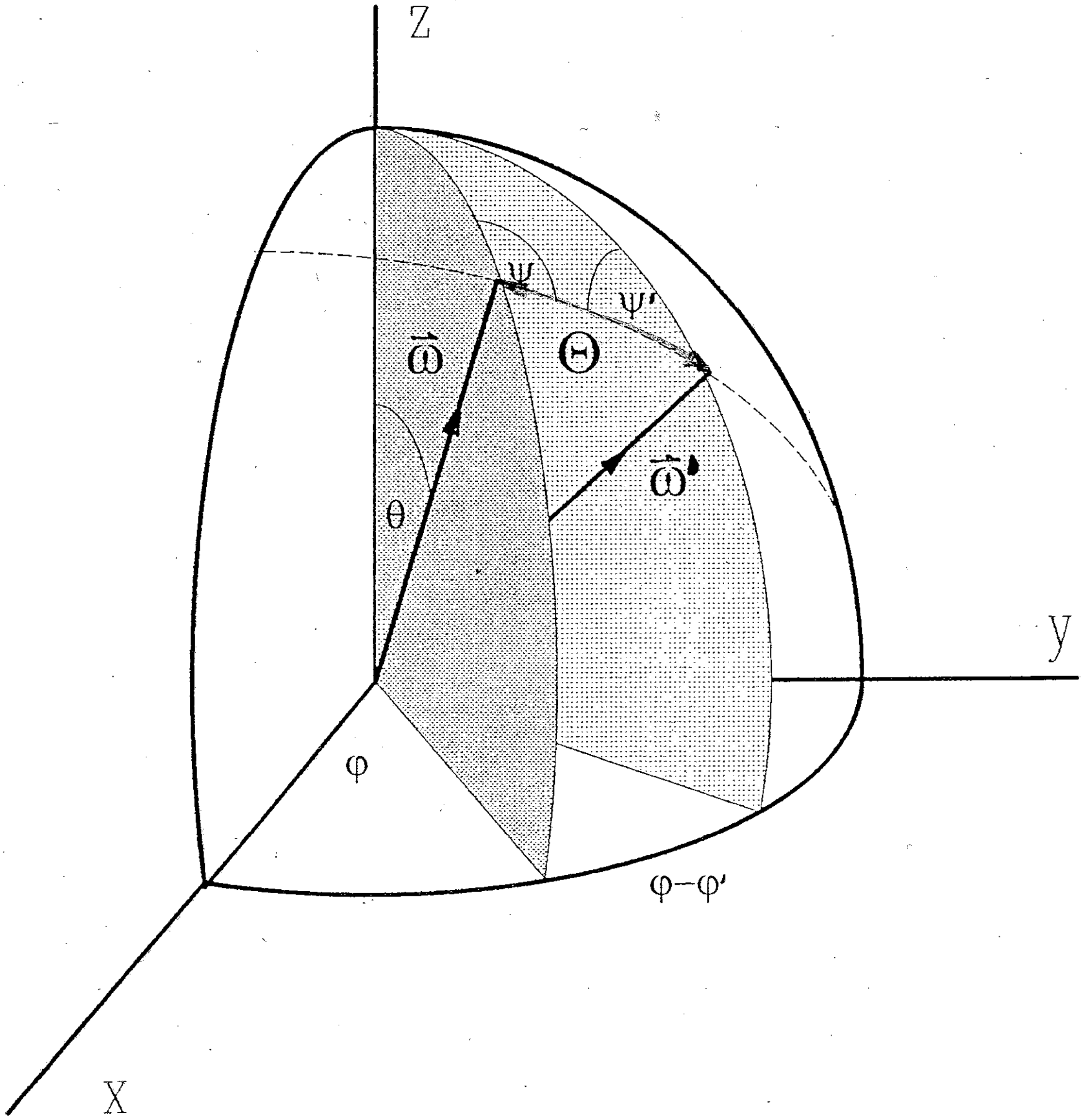
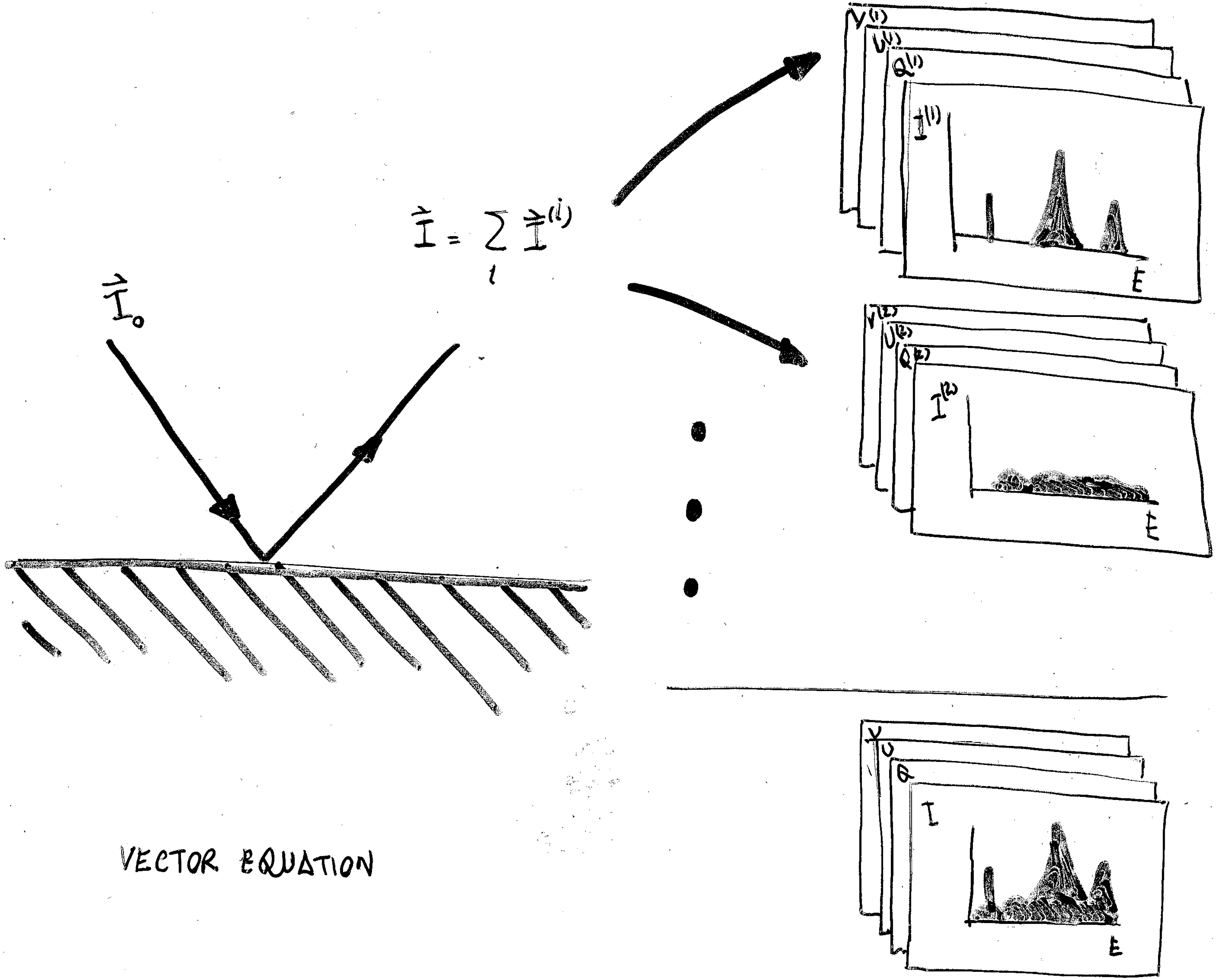


Figure 2. Transformation between the plane of scattering and the meridian plane.



VECTOR EQUATION

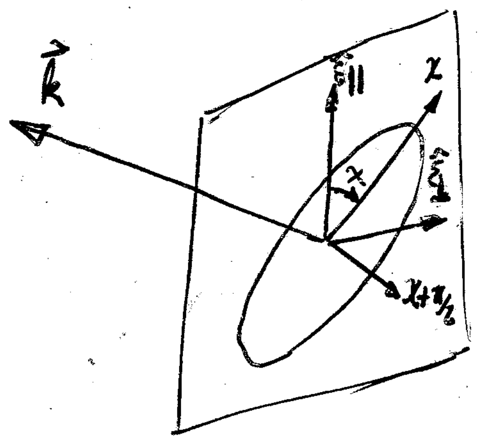
REPRESENTATION OF POLARIZED RADIATION (SUMMARY)

four parameters {
- STOKES parameters: (I, Q, U, V) "(S) system"
- Scattering parameters: $(I_{\parallel}, I_{\perp}, U, V)$ "(L) system"

$$I = I_{\parallel} + I_{\perp}$$
$$Q = I_{\parallel} - I_{\perp}$$

to specify {
- the intensity
- the degree of polarization
- the orientation of the ellipse of polarization
- the ellipticity

$$I = (E_{\parallel}^{0})^2 + (E_{\perp}^{0})^2$$
$$Q = (E_{\parallel}^{0})^2 - (E_{\perp}^{0})^2$$
$$U = 2 E_{\parallel}^{0} E_{\perp}^{0} \cos(\delta_{\parallel} - \delta_{\perp})$$
$$V = 2 E_{\parallel}^{0} E_{\perp}^{0} \sin(\delta_{\parallel} - \delta_{\perp})$$



$$Q = I \cos 2\beta \cos 2\chi$$
$$U = I \cos 2\beta \sin 2\chi$$
$$V = I \sin 2\beta$$

I represents the whole intensity

χ gives the rotation of the ellip. major axis about the coordinate system $\hat{\varepsilon}_{\parallel}, \hat{\varepsilon}_{\perp}$

β gives the ellipticity (major/minor axis ratio)

FULL ELLIPTICAL POLARIZATION

$$\sin 2\beta = \frac{I_V}{I_I}$$

$$\operatorname{tg} 2\chi = \frac{I_U}{I_Q}$$

and

$$I_I^2 = I_Q^2 + I_U^2 + I_V^2$$

PARTIAL ELLIPTICAL POLARIZATION

$$I_I^2 \geq I_Q^2 + I_U^2 + I_V^2$$

UNPOLARIZED BEAM

$$I_Q = I_U = I_V = 0$$

GENERAL BEAM

$$\sin 2\beta = \frac{I_V}{(I_Q^2 + I_U^2 + I_V^2)^{1/2}}$$

$$\operatorname{tg} 2\chi = \frac{I_U}{I_Q}$$

$$P = \frac{(I_Q^2 + I_U^2 + I_V^2)^{1/2}}{I_I}$$

$$\begin{pmatrix} I_I \\ I_Q \\ I_U \\ I_V \end{pmatrix} = I_I (1-P) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + I_I P \begin{pmatrix} 1 \\ \cos 2\chi \cos 2\beta \\ \sin 2\chi \cos 2\beta \\ \sin 2\beta \end{pmatrix}$$

TABLE I.

Some characteristic states of polarization are shown using the set of parameters S, normalized to an unitary total intensity. χ denotes the orientation of the line of polarization with respect to the intersection of the polarization plane with the scattering plane.

Polarization state	Set S (I,Q,U.V)
Unpolarized	(1,0,0,0)
Linear (generic)	$(1,\cos 2\chi,\sin 2\chi,0)$
Linear (∥) (parallel)	(1,1,0,0)
Linear (⊥) (perpendicular)	(1,-1,0,0)
Linear (45°)	(1,0,1,0)
Circular	(1,0,0,1)

KERNEL MATRICES

PHOTOELECTRIC EFFECT:

$$K_{P_{\lambda i}}(\overline{\omega},\lambda,\overline{\omega}',\lambda') = k_{P_{\lambda i}}(\overline{\omega},\lambda,\overline{\omega}',\lambda')\times\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{vmatrix}$$

$$k_{P_{\lambda i}}(\overline{\omega},\lambda,\overline{\omega}',\lambda') = \frac{1}{4\pi}Q_{\lambda_i}(\lambda')\delta(\lambda-\lambda_i)\times\left[1-U(\lambda'-\lambda_{ei})\right]$$

$$Q_{\lambda i}(\lambda') = \tau_s(\lambda')(1-1/J_{\lambda i})\omega_{\lambda i}\Gamma_{\lambda i}$$

RAYLEIGH SCATTERING:

$$k_R(\overline{\omega},\lambda,\overline{\omega}',\lambda') = \sigma\delta(\lambda-\lambda')\left(1+(\overline{\omega}\cdot\overline{\omega}')^2\right)\frac{F^2(\lambda',\overline{\omega}\cdot\overline{\omega}',Z)}{Z}$$

$$K_R = \frac{k_R(\overline{\omega},\lambda,\overline{\omega}',\lambda')}{1+(\overline{\omega}\cdot\overline{\omega}')^2} * \begin{vmatrix}1+(\overline{\omega}\cdot\overline{\omega}')^2 & \left[(\overline{\omega}\cdot\overline{\omega}')^2-1\right] & 0 & 0\\ \left[(\overline{\omega}\cdot\overline{\omega}')^2-1\right] & 1+(\overline{\omega}\cdot\overline{\omega}')^2 & 0 & 0\\ 0 & 0 & 2(\overline{\omega}\cdot\overline{\omega}') & 0\\ 0 & 0 & 0 & 2(\overline{\omega}\cdot\overline{\omega}')\end{vmatrix}$$

COMPTON SCATTERING (without Doppler):

$$k_C(\overline{\omega},\lambda,\overline{\omega}',\lambda') = \sigma K_{KN}(\lambda,\lambda')S(\lambda',\overline{\omega}\cdot\overline{\omega}',Z)\frac{1}{\lambda_C}\delta\left(1-\overline{\omega}\cdot\overline{\omega}'+\frac{\lambda'-\lambda}{\lambda_C}\right)$$

$$K_{KN}(\lambda,\lambda') = \left(\frac{\lambda'}{\lambda}\right)^2\left\{\frac{\lambda}{\lambda'}+\frac{\lambda'}{\lambda}+\frac{\lambda-\lambda'}{\lambda_C}\left(\frac{\lambda-\lambda'}{\lambda_C}-2\right)\right\}$$

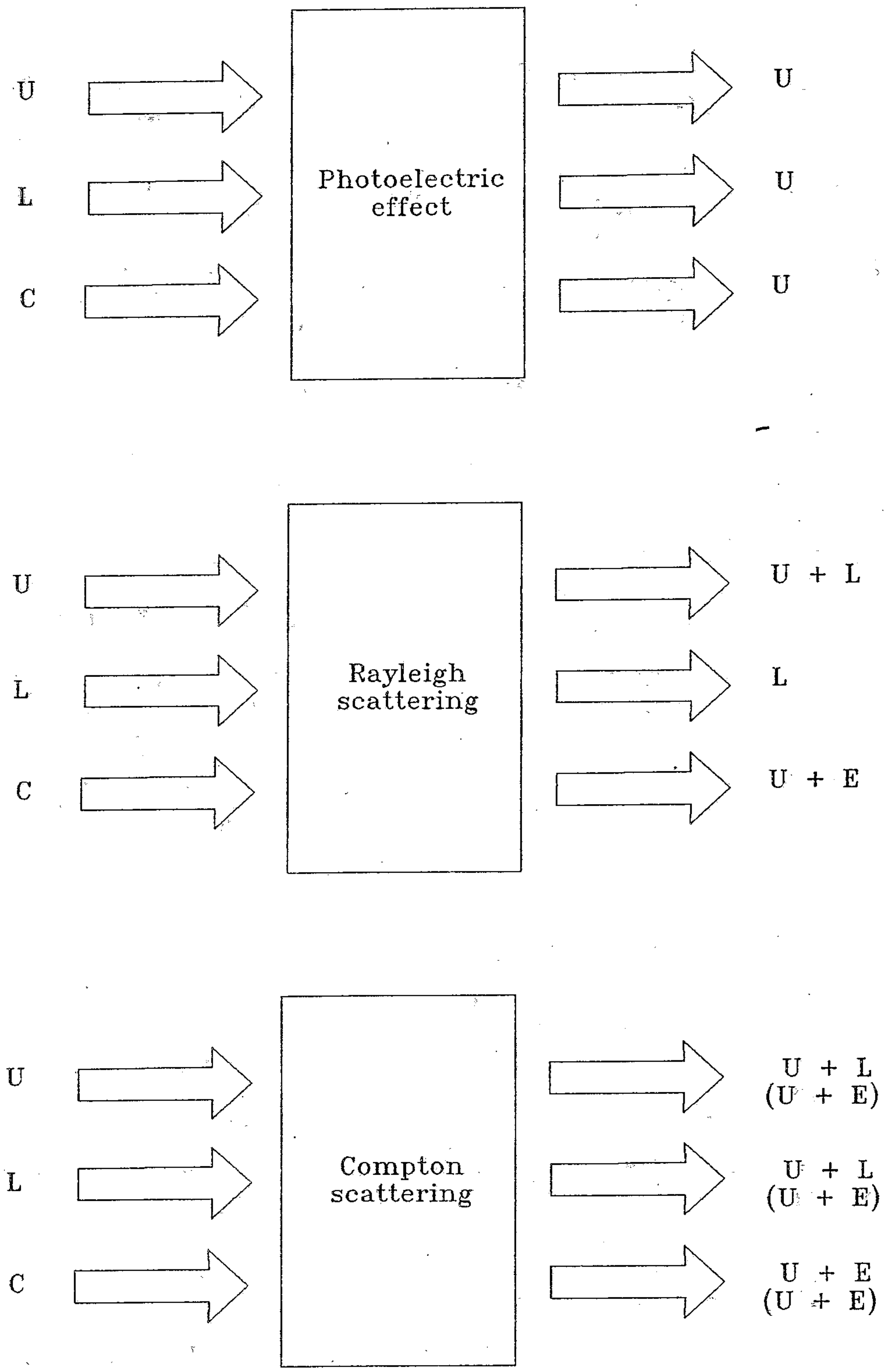
$$K_C(\overline{\omega},\lambda,\overline{\omega}',\lambda') = \frac{k_C(\overline{\omega},\lambda,\overline{\omega}',\lambda')}{a+b(b-2)} * \begin{vmatrix}a+b(b-2) & b(b-2) & 0 & 0\\ b(b-2) & \left[2+b(b-2)\right] & 0 & 0\\ 0 & 0 & (1-b) & 0\\ 0 & 0 & 0 & a(1-b)\end{vmatrix}$$

with

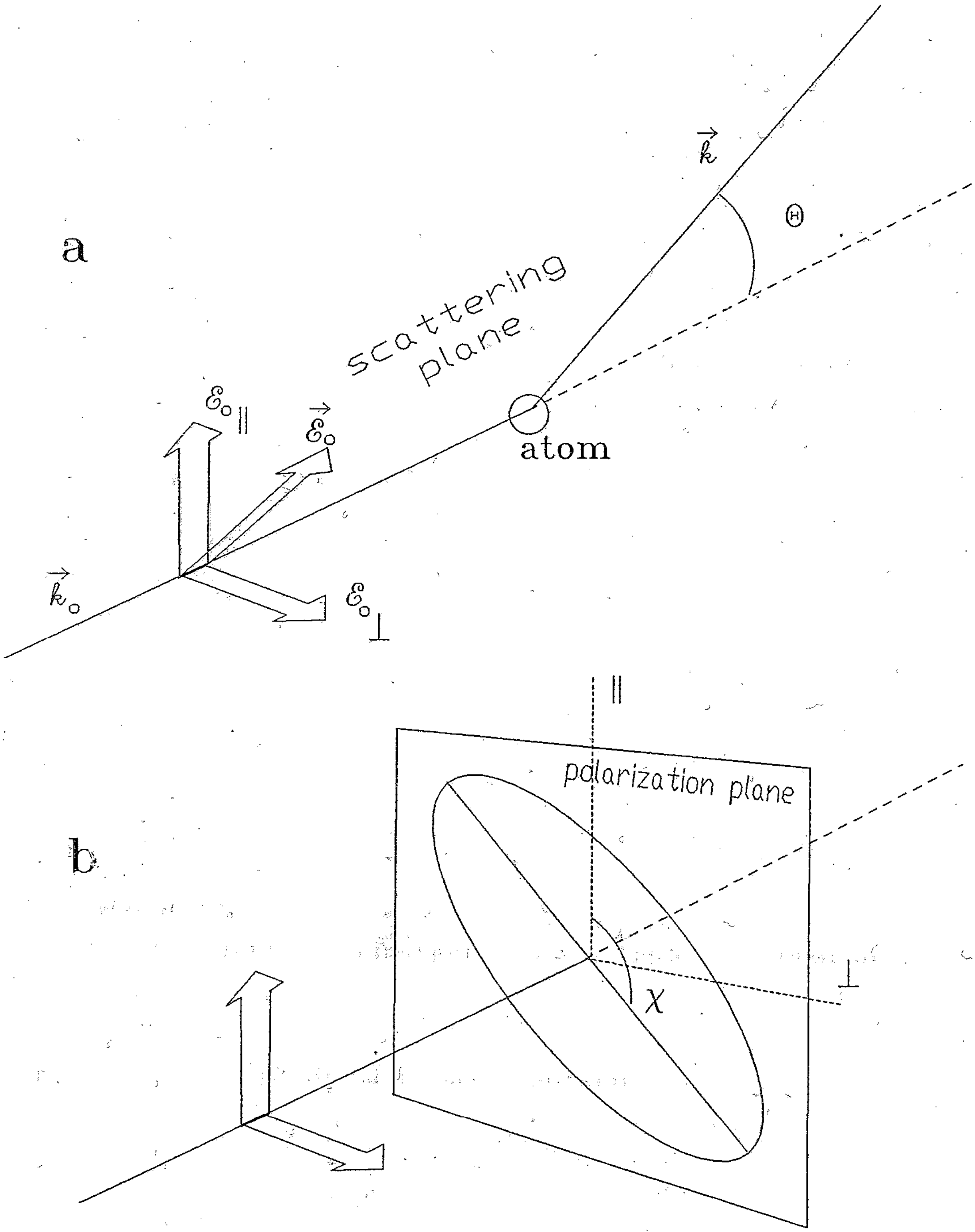
$$a = \frac{\lambda}{\lambda'}+\frac{\lambda'}{\lambda} = \frac{\left[\lambda'+\lambda_C(1-\overline{\omega}\cdot\overline{\omega}')\right]^2+\lambda'^2}{\lambda'\left[\lambda'+\lambda_C(1-\overline{\omega}\cdot\overline{\omega}')\right]},$$

$$b = \frac{\lambda-\lambda'}{\lambda_C}$$

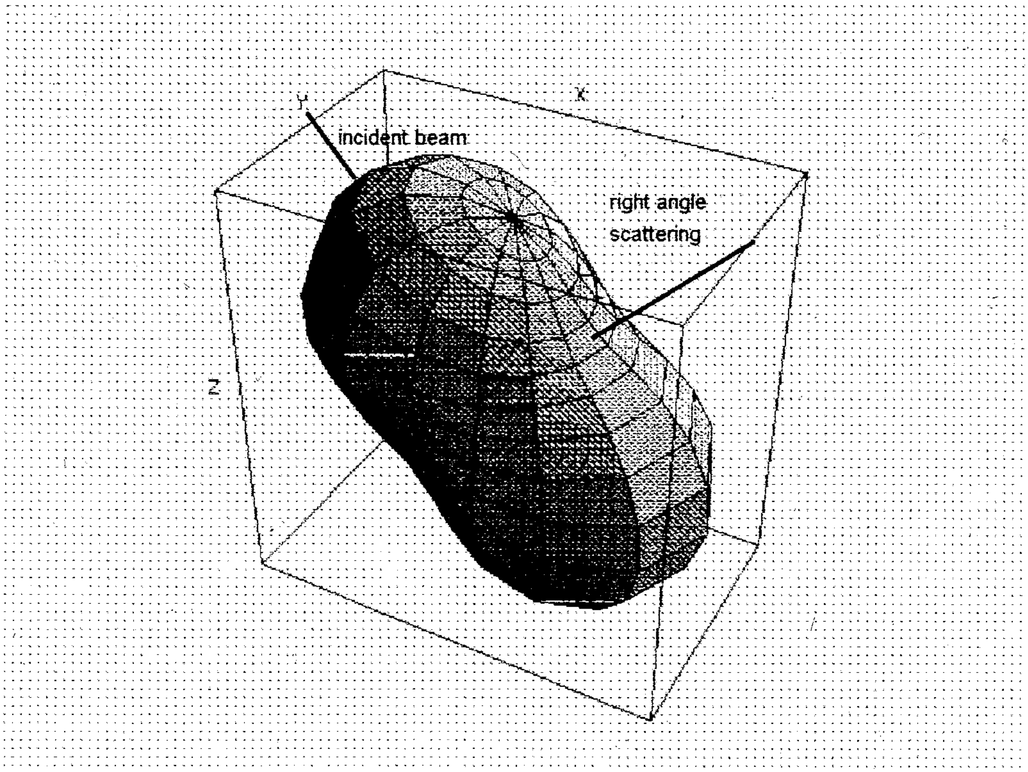
CHANGES IN THE POLARIZATION STATE PRODUCED BY DIFFERENT INTERACTIONS

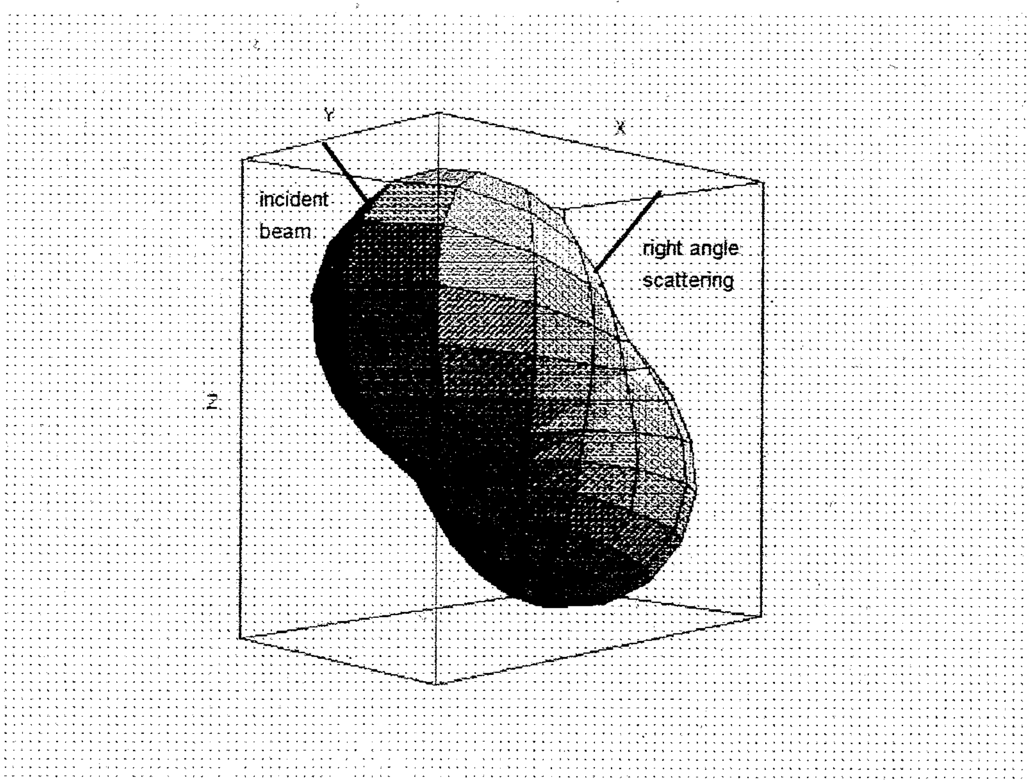


U: unpolarized; L: linearly polarized;
C: circularly polarized; E: elliptically polarized
Values between parenthesis correspond to an applied external magnetic field.

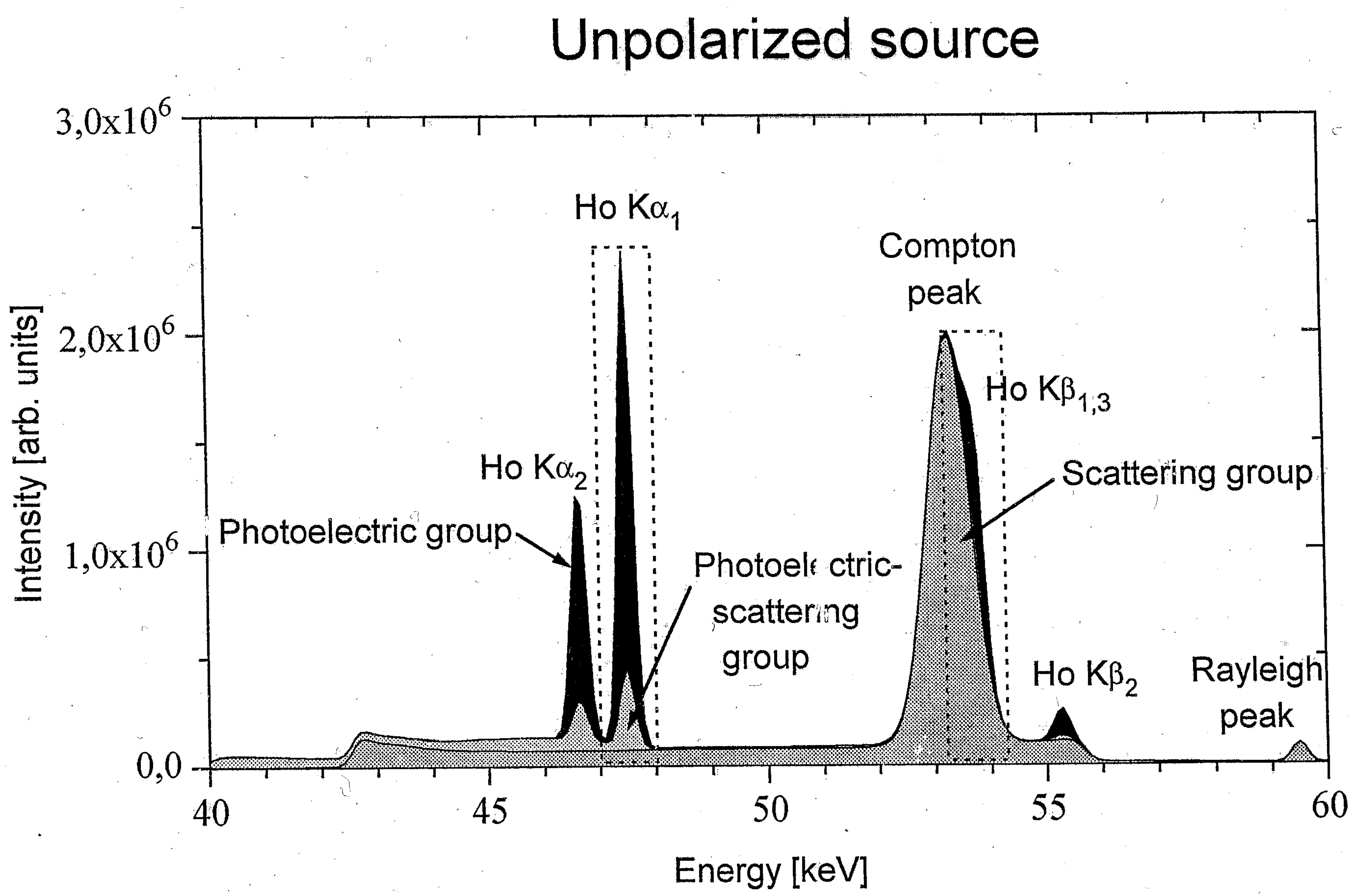

a
$\vec{k}$
Θ
scattering plane
atom
$\mathcal{E}_{o\parallel}$
$\vec{\mathcal{E}}_o$
$\vec{k}_o$
$\mathcal{E}_{o\perp}$
b
$\parallel$
polarization plane
χ
$\perp$

Source: Unpolarized



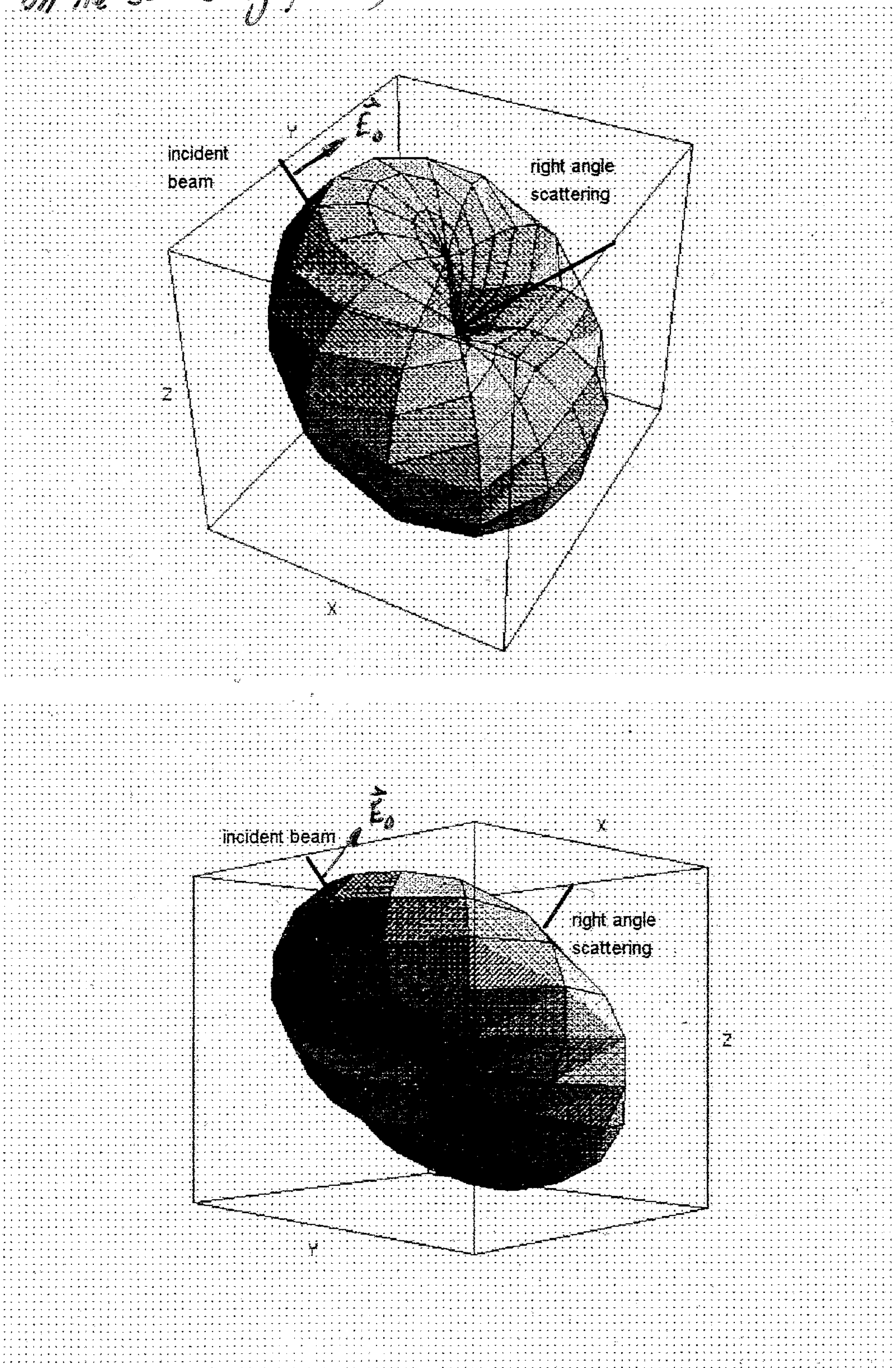


Angular function $\{1 + \cos^2\Theta\}/2$ in the first-order intensity term corresponding to an unpolarized source. The solid line on the left shows the incidence direction, while the solid line on the right illustrates a 90° scattering direction lying on the plane x-z. The strength of the angular function (here represented by the vector radius to the surface of the solid) is minimum and constant for 90° scattering in all directions.


Unpolarized source
3,0x10^6
2,0x10^6
1,0x10^6
0,0
Intensity [arb. units]
40
45
50
55
60
Energy [keV]
Ho Kα1
Ho Kα2
Photoelectric group
Photoelectric-scattering group
Compton peak
Ho Kβ1,3
Scattering group
Ho Kβ2
Rayleigh peak

Source = LP

$\chi = 0$ (on the scattering plane)



Angular function $\{1 - \sin^2\Theta \cos^2(\psi' + \chi)\}$ in the first-order intensity term corresponding to a linearly polarized source having the line of polarization oriented along the z-axis (i.e. $\chi = 0$, or the electric vector of the incident beam propagating in the plane x-z). The solid line on the left shows the incidence direction, while the solid line on the right illustrates a 90° scattering direction lying on the plane x-z. The strength of the angular function (here represented by the vector radius to the surface of the solid) is null for this direction. In contrast, it is maximum for the scattering vectors lying on a plane normal to the plane x-z which contains the incidence direction.

Linearly polarized source (χ=0