For $\underline{n=0}$ Eqns (6.a) and (6.b) give

$$iq\eta\,\tilde f_-^{(0)} = -\mu(\lambda)\,\tilde f_+^{(0)} + I_0\,\delta(\vec w-\vec w_0)\,\delta(\lambda-\lambda_0)$$

$$iq\eta\,\tilde f_+^{(0)} = -\mu(\lambda)\,\tilde f_-^{(0)}$$

from which

$$\tilde f_+^{(0)} = \frac{\mu}{\mu^2+q^2\eta^2}\, I_0\,\delta(\vec w-\vec w_0)\,\delta(\lambda-\lambda_0)$$

$$\tilde f_-^{(0)} = \frac{-iq\eta}{\mu^2+q^2\eta^2}\, I_0\,\delta(\vec w-\vec w_0)\,\delta(\lambda-\lambda_0)$$

The solution in the physical space is obtained by using the Fourier inversion formula for real functions

since the flux is real!

$$\underbrace{g(z)}_{\text{is real}} = \frac{1}{\pi}\,\mathrm{Re}\left\{\int_0^\infty dq\, e^{iqz}\,\tilde g(q)\right\}$$

Indeed, it can be shown that (HOMEWORK)

$$\boxed{f^{(0)}(z,\vec{w},\lambda) = \frac{I_0}{2|\eta|}\,\delta(\vec{w}-\vec{w}_0)\,\delta(\lambda-\lambda_0)\,e^{-\frac{\mu|z|}{|\eta|}}\,(1+\operatorname{sgn}\eta\,\operatorname{sgn}z)}$$

For $n>0$ Eqns (6) give

$$iq\eta\,\tilde{f}_-^{(n)} = -\mu(\lambda)\tilde{f}_+^{(n)} + \frac{1}{2}\hat{I}\tilde{f}_+^{(n-1)} - \frac{i}{2}\hat{I}\hat{k}\tilde{f}_-^{(n-1)}$$

$$iq\eta\,\tilde{f}_+^{(n)} = -\mu(\lambda)\tilde{f}_-^{(n)} + \frac{1}{2}\hat{I}\tilde{f}_-^{(n-1)} - \frac{i}{2}\hat{I}\hat{k}\tilde{f}_+^{(n-1)}$$

in matrix way

$$\underbrace{\begin{pmatrix} iq\eta & \mu \\ \mu & iq\eta \end{pmatrix}}_{\mathbb{A}}\underbrace{\begin{pmatrix} \tilde{f}_-^{(n)} \\ \tilde{f}_+^{(n)} \end{pmatrix}}_{\hat{P}\tilde{f}^{(n)}} = \frac{\hat{I}}{2}\underbrace{\begin{pmatrix} -i\hat{k} & 1 \\ 1 & -i\hat{k} \end{pmatrix}}_{\mathbb{K}}\underbrace{\begin{pmatrix} \tilde{f}_-^{(n-1)} \\ \tilde{f}_+^{(n-1)} \end{pmatrix}}_{\hat{P}\tilde{f}^{(n-1)}}$$

$$\Rightarrow \mathbb{A}\hat{\mathbb{P}}\tilde{f}^{n} = \frac{\hat{I}}{2}\mathbb{K}\hat{\mathbb{P}}\tilde{f}^{(n-1)} \qquad (7)$$

where

$$\hat{P}\tilde{f}^{(n)} = \begin{pmatrix} \tilde{f}_{-}^{(n)} \\ \tilde{f}_{+}^{(n)} \end{pmatrix}$$

projection on the even and odd parts

From Equ (7) we have

$$\boxed{\begin{aligned} \hat{P}\tilde{f}^{(n)} &= \mathbb{A}^{-1}\,\frac{\hat{I}}{2}\,\mathbb{K}\,\hat{P}\tilde{f}^{(n-1)} \\ &= \frac{\hat{I}}{2}\,\mathbb{A}^{-1}\mathbb{K}\,\hat{P}\tilde{f}^{(n-1)} \qquad (8) \end{aligned}}$$

By using recursively the last equation we found

$$\hat{P}\tilde{f}^{(n)} = 2^{-n}\left[\mathbb{A}^{-1}\mathbb{K}\,\hat{I}\right]^{n}\,\hat{P}\tilde{f}^{(0)} \qquad (9)$$

and finally we obtain for the total flux

$$\hat{P}\tilde{f} = \sum_{n=1}^{\infty} 2^{1-n}\left[\mathbb{A}^{-1}\mathbb{K}\hat{I}\right]^{n-1}\hat{P}\tilde{f}^{(0)}$$

In the physical space Eqn (9) can be written as

$$\hat{P}f^{(n)} = \frac{\hat{I}}{2} Re\left(\mathcal{F}^{-1}\left\{\mathbb{A}^{-1}\mathbb{K}\hat{P}\tilde{f}^{(n-1)}\right\}\right)$$

$$= \frac{\hat{I}}{2} Re\left[\mathcal{F}^{-1}(\mathbb{A}^{-1}) \otimes \mathcal{F}^{-1}(\mathbb{K}\hat{P}\tilde{f}^{(n-1)})\right]$$

$$= \frac{\hat{I}}{2} Re\left(\mathcal{F}^{-1}(\mathbb{A}^{-1}) \otimes \begin{pmatrix} \operatorname{sgn} z & 1 \\ 1 & \operatorname{sgn} z \end{pmatrix}\hat{P}f^{(n-1)}\right)$$

$\Rightarrow$ (HOMEWORK)

Note that the noncommutability of the matrix operator with the projection operator $\hat{P}$ means that the solution parity cannot be preserved at any order.

$$f^{(n)}(z,\vec{\omega},\lambda) = \frac{\hat{I}}{2|\eta|}\int_0^{\infty} d\tau\, e^{-\frac{|z-\tau|\mu}{|\eta|}} \times \left(1+\operatorname{sgn}\eta\,\operatorname{sgn}(z-\tau)\right) f^{(n-1)}(\tau,\vec{\omega}',\lambda') \qquad (10)$$

We can divide the integral in τ according to the sign of $z-\tau$ which gives different results depending on the sign of z:

POSITIVE z

$$f^{(n)}(z,\vec{\omega},\lambda)=\frac{1}{|\eta|}\left\{\frac{1+\mathrm{sgn}\,\eta}{2}\,e^{-\frac{z\mu}{|\eta|}}\int_0^z d\tau\, e^{\frac{\tau\mu}{|\eta|}}\,\hat{I}f^{(n-1)}(\tau,\vec{\omega}';\lambda')\right.$$

$$\left.+\ \frac{1-\mathrm{sgn}\,\eta}{2}\int_0^\infty d\tau\, e^{-\frac{\tau\mu}{|\eta|}}\,\hat{I}f^{(n-1)}(\tau+z,\vec{\omega}';\lambda')\right\} \qquad (11)$$

NEGATIVE z

$$f^{(n)}(z,\vec{\omega},\lambda)=\frac{1}{|\eta|}\,\frac{1-\mathrm{sgn}\,\eta}{2}\,e^{-\frac{|z|\mu}{|\eta|}}\int_0^\infty d\tau\, e^{-\frac{\tau\mu}{|\eta|}}\,\hat{I}f^{(n-1)}(\tau,\vec{\omega}';\lambda') \qquad (12)$$

For $z\to 0$ the flux given by Eqns (11) and (12) coincide $\longrightarrow$ ALBEDO FLUX

The intensity is given by (12')

$$I^{(n)}(\vec{\omega},\lambda)=|\eta|\,\underbrace{f^{(n)}(0,\vec{\omega},\lambda)}_{\text{albedo flux}}$$