Boltzmann equation for photons and solution

Jorge E. Fernández

Laboratory of Montecuccolino-DIN, Alma Mater Studiorum
University of Bologna, via dei Colli 16, 40136 Bologna, Italy

- Boltzmann equation for photons

- Development in orders of interaction

- Solution

- Interaction kernels

1D Boltzmann equation for photons

$$
\eta \frac{\partial f(z,\vec{\omega},\lambda)}{\partial z} = -\mu(\lambda) f(z,\vec{\omega},\lambda) + \int_0^{\infty} d\lambda' \int_{4\pi} d\vec{\omega}' \, \underline{u(z)} \, k(\vec{\omega},\lambda,\vec{\omega}';\lambda') f(z,\vec{\omega}';\lambda') + I_0 \, \delta(z)\, \delta(\vec{\omega}-\vec{\omega}_0)\, \delta(\lambda-\lambda_0) \qquad (2)
$$

Heaviside step function

$$
u(z) = \begin{cases} 0 & z<0 \\ 1 & z>0 \end{cases}
$$

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Development in orders of interaction (Newman series)

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \dots + f^{(n)} + \dots$$

where

$f^{(n)}$ is the n-th order flux, i.e. the flux due to a chain of n interactions

$$\eta \frac{\partial f^{(n)}(z,\vec{\omega},\lambda)}{\partial z} = -\mu(\lambda) f^{(n)}(z,\vec{\omega},\lambda) +$$

$$+ \int_0^\infty d\lambda' \int_{4\pi} d\vec{\omega}'\, k(\vec{\omega}',\lambda',\vec{\omega};\lambda)\, U(z)\, f^{(n-1)}(z,\vec{\omega}';\lambda')\,[1-\delta_{n0}]$$

$$+ I_0\, \delta(z)\, \delta(\vec{\omega}-\vec{\omega}_0)\, \delta(\lambda-\lambda_0)\, \delta_{n0} \qquad (n = 0,1,2,\dots)$$

(2)

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Solution

$$f^{(0)}(z,\vec{\omega},\lambda) = \frac{I_0}{2|\eta|}\,\delta(\vec{\omega}-\vec{\omega}_0)\,\delta(\lambda-\lambda_0)\,e^{-\frac{\mu|z|}{|\eta|}}\,(1+\operatorname{sgn}\eta\,\operatorname{sgn} z)$$

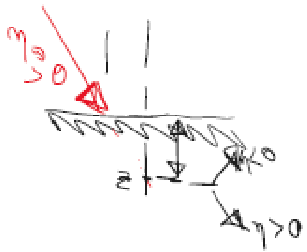
$$f^{(n)}(z,\vec{\omega},\lambda) = \frac{\hat{I}}{2|\eta|}\int_0^{\infty} d\tau\, e^{-\frac{\boxed{z-\tau}\,\mu}{|\eta|}}\,(1+\operatorname{sgn}\eta\,\operatorname{sgn}(z-\tau))\,f^{(n-1)}(\tau,\vec{\omega}',\lambda')$$

where $$\hat{I} = \int_0^{\infty} d\lambda' \int_{4\pi} d\vec{\omega}'\, k(\vec{\omega},\lambda,\vec{\omega}';\lambda')$$

The intensity is given by

$$I^{(n)}(\vec{\omega},\lambda) = \underbrace{|\eta| f^{(n)}(0,\vec{\omega},\lambda)}_{\text{albedo flux}}$$

Example: Computation for n=1



$$1+\operatorname{sgn}\eta\,\operatorname{sgn}(z-\tau)$$

$\tau \to [0,z]$ >0

$\tau \to [z,\infty]$ <0

$$f^{(0)}(z,\bar\omega',\lambda') = \frac{I_0}{2|\eta_0|}\,\delta(\bar\omega'-\bar\omega_0)\,\delta(\lambda'-\lambda_0)\,e^{-\frac{\mu(\lambda_0)}{|\eta_0|}|z|}$$

$$f^{(1)}(z,\bar\omega,\lambda) = \frac{\hat I}{2|\eta|}\int_0^\infty d\tau\, e^{-\frac{\mu(\lambda)}{|\eta|}|z-\tau|}\,(1+\operatorname{sgn}\tau\,\operatorname{sgn}\eta_0)\,f^{(0)}(\tau,\bar\omega',\lambda')$$

$$(1+\operatorname{sgn}\eta\,\operatorname{sgn}(z-\tau))$$

$$= \frac{I_0}{2|\eta||\eta_0|}\int_0^\infty d\tau\, e^{-\frac{\mu(\lambda)}{|\eta|}|z-\tau|}\,e^{-\frac{\mu_0}{|\eta_0|}|\tau|}\,k(\bar\omega_0,\lambda_0,\bar\omega,\lambda)$$

$$(1+\operatorname{sgn}\eta\,\operatorname{sgn}(z-\tau))$$

$$f^{(1)}(0,\bar\omega,\lambda) = \frac{I_0}{\cancel{2}|\eta||\eta_0|}\,\cancel{2}\int_0^\infty d\tau\, e^{-\left(\frac{\mu}{|\eta|}+\frac{\mu_0}{|\eta_0|}\right)|\tau|}\,k(\bar\omega_0,\lambda_0,\omega,\lambda)$$

$$= \frac{I_0}{|\eta||\eta_0|}\,\frac{1}{\frac{\mu}{|\eta|}+\frac{\mu_0}{|\eta_0|}} = \frac{I_0\,\boxed{k(\omega_0,\lambda_0,\omega,\lambda)}}{\mu|\eta_0|+\mu_0|\eta|}$$

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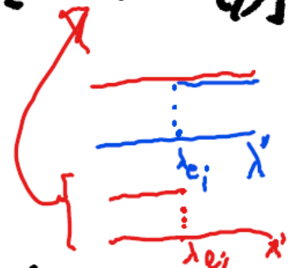
Kernel for the photoelectric effect

Photoelectric kernel

Isotropic (since radiative emission is an spontaneous process separated from photoelectric)

$$k_p(\omega,\lambda,\vec{\omega};\lambda') = \frac{1}{4\pi}\sum_i Q_{\lambda_i}(\lambda')\,\delta(\lambda-\lambda_i)\left[1-U(\lambda'-\lambda_{e_i})\right]$$

$$Q_{\lambda_i}(\lambda') = W_s\,\overbrace{\tau_s(\lambda')\left(1-\frac{1}{r_{\lambda_i}}\right)}^{\tau_{K_s}(\lambda')}\,\omega_{\lambda_i}\,\Gamma_{\lambda_i}$$

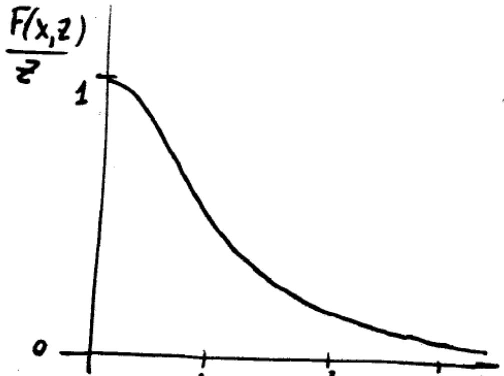


W_s: Weight fraction of element s

Γ_{λ_i}: intensity fraction of the line i in its own series

Kernel for Rayleigh scattering

$$k_R(\vec{\omega}, \lambda, \vec{\omega}', \lambda') = \underbrace{\delta(\lambda - \lambda')}_{\text{scatt. elastico}} \underbrace{\frac{r_e^2}{2}\left(1 + (\vec{\omega}\cdot\vec{\omega}')^2\right)}_{\text{Thompson differential cross section}} \underbrace{F^2(\lambda', \vec{\omega}\cdot\vec{\omega}', Z)}_{\text{form factor (shielding)}}$$



$$x = \frac{\sin(\theta/2)}{\lambda} = \frac{E}{hc}\sin(\theta/2)$$

Kernel for Compton Scattering

$$k_c(\hat{w},\lambda,\hat{w}';\lambda') = \underbrace{\frac{r_e^2}{2} K_{KN}(\lambda,\lambda')}_{\text{Klein-Nishina electronic differential cross section}} \underbrace{S(\lambda', \hat{w}\cdot\hat{w}', Z)}_{\text{Scattering function}} \, \delta\underbrace{\left(\lambda'-\lambda+\lambda_c(1-\hat{w}\cdot\hat{w}')\right)}_{\text{Compton shift}}$$

where

$$K_{KN}(\lambda,\lambda') = \left(\frac{\lambda'}{\lambda}\right)^2 \left\{\frac{\lambda}{\lambda'}+\frac{\lambda'}{\lambda}+\frac{\lambda-\lambda'}{\lambda_c}\left(\frac{\lambda-\lambda'}{\lambda_c}-2\right)\right\}$$

$$\lambda'-\lambda = \underbrace{\boxed{\left(\frac{h}{m_e c}\right)}}_{\lambda_c}(1-\cos\theta)$$

Compton wavelength

Compton shift

Kernel for Compton Scattering (cont.)

$$E_p = \frac{E_0}{1+\frac{E_0}{mc^2}(1-\cos\theta)} = \frac{E_0}{1+\alpha(1-\cos\theta)}$$

with $\alpha = \frac{E_0}{mc^2}$.

$$\left(\frac{d\sigma}{d\Omega}\right)_{KN} = \int_0^\infty \frac{1}{2} r_e^2 KN(E_0, E, \theta)\delta\left(E - E_p\right)dE$$

$$= \frac{1}{2} r_e^2 KN\left(E_0, E_p, \theta\right)$$

$$= \frac{1}{2} r_e^2 \left(\frac{1}{1+\alpha(1-\cos\theta)}\right)^2 \left(1 + \cos^2\theta + \frac{\alpha^2(1-\cos\theta)^2}{1+\alpha(1-\cos\theta)}\right)$$

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Thank you for your attention!

jorge.fernandez@unibo.it