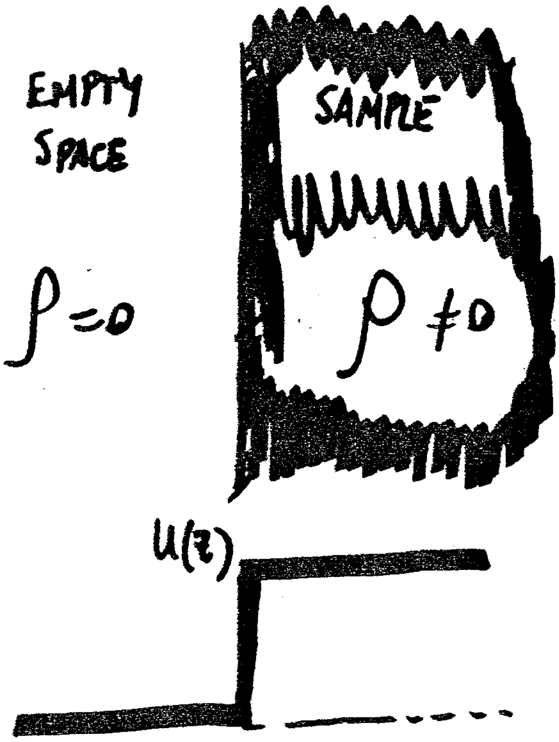
We need to consider a semi-infinite space

$$\eta \frac{\partial f(z,\vec{\omega},\lambda)}{\partial z} = -\mu(\lambda) f(z,\vec{\omega},\lambda) + \int_0^\infty d\lambda' \int_{4\pi} d\vec{\omega}' \, \underline{U(z)} \, k(\vec{\omega},\lambda,\vec{\omega}';\lambda') f(z,\vec{\omega}';\lambda') + I_0 \, \delta(z) \, \delta(\vec{\omega}-\vec{\omega}_0) \, \delta(\lambda-\lambda_0) \qquad (2)$$

Heaviside step function

$$U(z) = \begin{cases} 0 & z<0 \\ 1 & z>0 \end{cases}$$

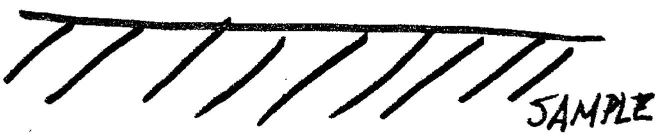


- empty space does not restitute photons (Photons going left does not return.
- Boltzmann Eqn remains unchanged
- $\mu(\lambda)$ remains unchanged

The change in the density ρ <u>is not</u> a requirement.

The properties of the physical model are completely defined by the <u>non-restitution</u> of the upper half-space.

$\rho \neq 0$ only absorbing medium



$\rho \neq 0$ absorbing and interacting medium

Absence of interactions in the upper semi-space prevents the return of photons escaping in that direction.

<u>Advantage</u>: μ <u>does not</u> change

DEVELOPMENT IN ORDERS OF INTERACTION (Neumann Series)

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \dots + f^{(n)} + \dots$$

where

$f^{(n)}$ is the n-th order flux, i.e. the flux due to a chain of n interactions

By substitution in (2) we obtain an equation set

$$\eta \frac{\partial f^{(n)}(z,\vec{\omega},\lambda)}{\partial z} = -\mu(\lambda) f^{(n)}(z,\vec{\omega},\lambda) + $$
$$+ \int_0^\infty d\lambda' \int_{4\pi} d\vec{\omega}'\, k(\vec{\omega},\lambda,\vec{\omega}';\lambda')\, \mathcal{U}(z)\, f^{(n-1)}(z,\vec{\omega}';\lambda')\, [1-\delta_{n0}]$$
$$+ I_0\, \delta(z)\, \delta(\vec{\omega}-\vec{\omega}_0)\, \delta(\lambda-\lambda_0)\, \delta_{n0} \qquad (n = 0, 1, 2, \dots) \qquad (3)$$

which is the equation to be solved.

METHOD OF SOLUTION FOR Eq (3)

We split the flux into its even and odd parts

$$f_+(z) = \frac{1}{2}\left(f(z) + f(-z)\right) \quad \text{(even)}$$

$$f_-(z) = \frac{1}{2}\left(f(z) - f(-z)\right) \quad \text{(odd)}$$

In order to do this we write the set of equations (3) for $-z$:

$$\boxed{\begin{aligned} -\eta \frac{\partial f^{(m)}(-z,\vec{\omega},\lambda)}{\partial z} &= -\mu(\lambda) f^{(m)}(-z,\vec{\omega},\lambda) + \\ &+ \int_0^\infty d\lambda' \int_{4\pi} d\hat{\omega}'\, k(\vec{\omega},\lambda,\hat{\omega}';\lambda')\, \mathcal{U}(-z) f^{(m-1)}(-z,\vec{\omega}';\lambda') \\ &\qquad [1-\delta_{m0}] \\ &+ I_0\, \delta(-z)\,\delta(\vec{\omega}-\vec{\omega}_0)\,\delta(\lambda-\lambda_0)\,\delta_{m0} \qquad (4) \end{aligned}}$$

By the delta property

$$\delta(ax) = \frac{1}{|a|}\delta(x) \Rightarrow \delta(z) = \delta(-z)$$

and

$$\mathcal{U}(z) = \frac{1}{2}\left(1 + \operatorname{sgn} z\right)$$

$$\operatorname{sgn} z = \begin{cases} 1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases}$$

and $\operatorname{sgn}(-z) = -\operatorname{sgn} z$

So

$$U(-z) = \frac{1}{2}\left(1 - \operatorname{sgn} z\right)$$

By adding and substracting Eqns (3) and (4) and using the above properties we get

(3) + (4) $\Rightarrow$

$$\eta \frac{\partial}{\partial z}\left(f^{(m)}(+z) - f^{(m)}(-z)\right) = -\mu(\lambda)\left(f^{(m)}(z) + f^{(m)}(-z)\right) +$$

$$+ \left\{ \int_0^\infty d\lambda' \int_{4\pi} d\vec{w}'\, k(\vec{w}, \lambda, \vec{w}'; \lambda') \frac{\left[f^{(m-1)}(z) + f^{(m-1)}(-z)\right]}{2} \; + \right.$$

$$\left. + \operatorname{sgn} z \int_0^\infty d\lambda' \int_{4\pi} d\vec{w}'\, k(\vec{w}, \lambda, \vec{w}'; \lambda') \frac{\left[f^{(m-1)}(z) - f^{(m-1)}(-z)\right]}{2} \right\} \left[1 - \delta_{m0}\right]$$

$$+ \; 2\, I_0\, \delta(z)\, \delta(\vec{w} - \vec{w}_0)\, \delta(\lambda - \lambda_0)\, \delta_{m0}$$

which can be written as

$$\eta \frac{\partial}{\partial z} f_-^{(n)} = -\mu(\lambda) f_+^{(n)} +$$

$$+ \frac{1}{2} \Big\{ \int_0^\infty d\lambda' \int_{4\pi} d\vec{w}'\, k(\vec{w},\lambda,\vec{w}';\lambda')\, f_+^{(n-1)}(z,\vec{w}',\lambda') +$$

$$+ \operatorname{sgn} z \int_0^\infty d\lambda' \int_{4\pi} d\vec{w}'\, k(\vec{w},\lambda,\vec{w}';\lambda')\, f_-^{(n-1)}(z,\vec{w}',\lambda') \Big\} [1-\delta_{n0}]$$

$$+ I_0\, \delta(z)\, \delta(\vec{w}-\vec{w}_0)\, \delta(\lambda-\lambda_0)\, \delta_{n0}$$

We denote as $\hat{I}$ the <u>operator</u>

$$\hat{I} = \int_0^\infty d\lambda' \int_{4\pi} d\vec{w}'\, k(\vec{w},\lambda,\vec{w}';\lambda')$$

and then we can rewrite the above eqn as follows

$$\boxed{\begin{aligned} \eta \frac{\partial}{\partial z} f_-^{(n)} &= -\mu(\lambda) f_+^{(n)} + \\ &+ \Big\{ \frac{1}{2} \hat{I} f_+^{(n-1)} + \frac{1}{2} \hat{I} \operatorname{sgn} z\, f_-^{(n-1)} \Big\} (1-\delta_{n0}) \\ &+ I_0\, \delta(z)\, \delta(\vec{w}-\vec{w}_0)\, \delta(\lambda-\lambda_0)\, \delta_{n0} \qquad (5.a) \end{aligned}}$$

Similarly (3)-(4) gives (HOMEWORK)

$$\eta \frac{\partial f_+^{(m)}}{\partial z} = -\mu(\lambda) f_-^{(m)} + \left\{ \frac{1}{2}\hat{I} f_-^{(m-1)} + \frac{1}{2}\hat{I}\,\mathrm{sgn}\,z\, f_+^{(m-1)} \right\}(1-\delta_{m0}) \qquad (5.6)$$

The operator $\hat{I}$ commutes with functions of z (only)

$$\hat{I}\, g(z) = g(z)\, \hat{I}$$

but not with $f_+^{(m)}$ or $f_-^{(m)}$.

We can <u>Fourier transform</u> Eqns (5.a) and (5.b)

Definition

Fourier transform of a function $g(z)$

$$\mathcal{F}(g(z)) = \tilde{g}(q)$$

$$= \int_{-\infty}^{\infty} dz\, e^{-iqz}\, g(z)$$

Fourier transform gives real and imaginary parts, then

$$\tilde{g}(q) = Re(\tilde{g}) + i\, Im(\tilde{g})$$

$$\boxed{Re(\tilde{g}) = \tfrac{1}{2}\mathcal{F}(g_+)} \quad \text{and} \quad \boxed{i\,Im(\tilde{g}) = \tfrac{1}{2}\mathcal{F}(g_-)}$$

to give

$$i q \eta \tilde{f}_-^{(n)} = -\mu(\lambda) \tilde{f}_+^{(n)} +$$

$$+ \left\{ \frac{1}{2} \hat{I} \tilde{f}_+^{(n-1)} + \frac{1}{2} \hat{I} \mathcal{F}\left[\operatorname{sgn} z \, f_-^{(n-1)} \right] \right\} (1 - \delta_{n0})$$

$$+ I_0 \, \delta(\vec{w} - \vec{w}_0) \, \delta(\lambda - \lambda_0) \, \delta_{n0}$$

we have used the properties of the Fourier Transform

$$\mathcal{F}\left(\frac{\partial g}{\partial z}\right) = i q \tilde{g}$$

$$\mathcal{F}(\delta(z)) = 1$$

Still we can use a convolution property (in frequency)

$$\mathcal{F}(g_1(z) \, g_2(z)) = \tilde{g}_1 \otimes \tilde{g}_2$$

$$\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}_1(y) \, \tilde{g}_2(q-y) \, dy$$

in the case of above we get

$$\mathcal{F}\Big[\underbrace{\operatorname{sgn} z}_{g_2} \, \underbrace{f_-^{(n-1)}}_{g_1} \Big] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\operatorname{sgn} z) \Big|_{q-y} \, \mathcal{F}\left(f_-^{(n-1)}\right) \Big|_{y} \, dy$$

$$= \frac{1}{\cancel{2}\pi} \int_{-\infty}^{\infty} dy \frac{2}{i(q-y)} \tilde{f}_-^{(m-1)}$$

$$= -\frac{i}{\pi} \int_{-\infty}^{\infty} dy \frac{1}{(q-y)} \tilde{f}_-^{(m-1)}$$

$$\boxed{\mathcal{F}(\mathrm{sgn}\, z) = \frac{2}{iq}}$$

Fourier transform of (5.a) finally gives

$$\boxed{\begin{aligned} iq\eta\, \tilde{f}_-^{(m)} &= -\mu(\lambda) \tilde{f}_+^{(m)} + \\ &+ \left[\frac{1}{2} \hat{I} \tilde{f}_+^{(m-1)} - \frac{i}{2} \hat{I} \hat{k} \tilde{f}_-^{(m-1)}\right](1-\delta_{m0}) \\ &+ I_0\, \delta(\vec{\omega}-\vec{\omega}_0)\delta(\lambda-\lambda_0)\, \delta_{m0} \qquad (6.a) \end{aligned}}$$

where we have defined

$$\hat{K} = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \frac{1}{(q-y)}$$

Transformation of (5.b) gives (HOMEWORK)

$$\boxed{iq\eta\, \tilde{f}_+^{(m)} = -\mu(\lambda) \tilde{f}_-^{(m)} + \left[\frac{1}{2} \hat{I} \tilde{f}_-^{(m-1)} - \frac{i}{2} \hat{I} \hat{k} \tilde{f}_+^{(m-1)}\right] [1-\delta_{m0}] \qquad (6.b)}$$